The Magnetic Field of a Finite Solenoid

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Abstract

The magnetic field of a solenoid is an elementary problem in classical electrodynamics taught to undergraduate physics students. In this paper, we derive 7 approximations of the components of the magnetic field both within and outside a finite solenoid. These approximations can be used in practical application. More importantly, they can be used to strengthen the symmetry arguments regarding the magnetic field of an infinite solenoid.

Contents

1 Introduction

The magnetic field of a solenoid is an elementary problem in classical electrodynamics taught to undergraduate physics students. In this paper, we derive 7 approximations of the components of the magnetic field both within and outside a finite solenoid. These approximations can be used in practical application. More importantly, they can be used to strengthen the symmetry arguments regarding the magnetic field of an infinite solenoid.

2 Magnetic Field within a finite solenoid:

In the following derivations, we will use a cylindrical coordinate system with the standard (r,ϕ, z) coordinates. The z-axis lies on the central axis of the solenoid. ϕ is the azimuthal angle while r is measured from the center axis.
We will derive multiple approximations depending on the relative values of r’, R, and Len for a single component.

2.1 Axial component of magnetic field

To calculate the B-field on the axis of the solenoid, we utilize the well-known fact that the B field on the center axis of a current loop is given by:

$$B(loop) = \frac{\mu I R^2}{2(z^2+R^2)^{\frac{3}{2}}} \quad (1)$$

Where I is the current in the loop, R is its radius, and z is the distance from the center of the loop to any arbitrary point on the center axis. Assume a vacuum permeability constant of μ.
Assuming that the solenoid is a tightly-packed collection of loops, we can say that each loop contributes a

$$dB(loop) = \frac{(\mu)(dI)R^2}{2(z^2+R^2)^{\frac{3}{2}}} \quad (2)$$

to the total B-field on the axis of the solenoid.
For a given dz of the solenoid, the value of dI within that dz is given by dI=$Indz$, where $n = \frac{N}{Len}$, the number of turns (N) per unit length (Len). We then have:

$$dB(loop) \frac{(\mu I n R^2)dz}{2(z^2+R^2)^{\frac{3}{2}}} \quad (3)$$

Integration of this expression over the solenoid yields the total B_z component at any point P with z-coordinate z’ (measured from the center of the solenoid). This integral is easily solved by performing a standard trigonometric substitution with $z = R\tan\varnothing$.

$$B_z = \int_{-\frac{Len}{2}-z'}^{\frac{Len}{2}-z'} \frac{(\mu I n R^2)dz}{2(z^2+R^2)^{\frac{3}{2}}} = \frac{\mu n I}{2}\left(\frac{\frac{Len}{2}-z'}{\sqrt{R^2+\left(\frac{Len}{2}+z'\right)^2}} + \frac{\frac{Len}{2}+z'}{\sqrt{(R^2+\left(\frac{Len}{2}+z'\right)^2}}\right) \quad (4)$$

Of course, the above equation is only an approximation for a "real" solenoid, as it assumes that the solenoid is an infinitely tightly-packed stack of closed loops.

Normalizing this expression by substituting $K = \frac{Len}{R}$ and $C = \frac{z'}{R}$ and simplifying the resulting expressions gives:

$$B_z = \frac{\mu m I}{2}\left(\frac{\frac{K}{2}-C}{\sqrt{1+\left(\frac{K}{2}-C\right)^2}} + \frac{\frac{K}{2}+C}{\sqrt{\left(1+\left(\frac{K}{2}+C\right)^2\right.}}\right) \quad (5)$$

It is clear that $\lim_{K\to\infty} B_z = \mu m I$, the standard result given by most undergraduate textbooks. It is important to note that this holds as $Len \to \infty$ and R is finite, or $R \to 0$ and Len is finite.

2.2 Radial component of magnetic field

To get an approximation of the radial field within the solenoid, we approximate each loop as a square loop made up of 4 wires, each of length 2R.

$$B(wire) = \frac{\mu I}{4\pi r}\left(\frac{Len + 2z'}{\sqrt{4r^2 + (Len + 2z')^2}} + \left(\frac{Len - 2z')}{\sqrt{4r^2 + (Len - 2z')^2}}\right) \quad (30)$$

Where r is the radial distance from the wire to an arbitrary point p and z' is the distance from the center of the wire to P. r' denotes the perpendicular distance from P to the axis of the solenoid. We label the wire above P as wire "T", the opposite wire as wire "B", and the wire to the left looking down the axis as wire "1", while the opposite wire is wire "2". We place P an arbitrary distance z from the square loop

For wire T, $r = \sqrt{(R-r)^2 + z^2}$, z'=0, and Len=2R.

$$B(T) = \frac{\mu I R}{2\pi\sqrt{(R-r)^2 + z^2}}\left(\frac{1}{\sqrt{(R-r)^2 + z^2 + R^2)}}\right) \quad (1)$$

For wire B, $r = \sqrt{(R+r)^2 + z^2}$, z'=0, and Len=2R.

$$B(B) = \frac{\mu I R}{2\pi\sqrt{(R+r)^2 + z^2}}\left(\frac{1}{\sqrt{(R+r)^2 + z^2 + R^2)}}\right) \quad (2)$$

For wires 1 and 2, $r = \sqrt{R^2 + z^2}$, z'=r, and Len=2R.

$$B(1) = B(2) = \frac{\mu I}{4\pi\sqrt{R^2 + z^2}}\left(\frac{R + r}{\sqrt{z^2 + R^2 + (R+r)^2}} + \left(\frac{R - r}{\sqrt{z^2 + R^2 + (R-r)^2}}\right) \quad (3)$$

Vector components tell us that $|\mathbf{B}(T)|\cos\theta = |\mathbf{B_r}(T)|$, where $\mathbf{B_r}(T)$ is the radial component of $\mathbf{B}$(T) at P.
$\cos\theta$ is the angle between the z − axis and the position vector from P to the center of wire T. Here, $\cos\theta = \frac{z}{\sqrt{z^2+(R-r)^2}}$ P.

We also know that $|\mathbf{B}(B|)\sin\phi = |\mathbf{B_r}(B)|$, where $\mathbf{B_r}(B)$ is the radial component of $\mathbf{B}$(B) at P.
$\sin\phi$ is the angle between the axis orthogonal to the z − axis and the position vector from P to the center of wire B. Here, $\sin\varphi = \frac{z}{\sqrt{z^2+(R+r)^2}}$

Wires 1 and 2 do not contribute to $\mathbf{B_r}$ at P. Since $\mathbf{B_r}(T)$ and $\mathbf{B_r}(B)$ point in opposite directions, we have that

$$\mathbf{B_r} = \mathbf{B_r}(T) - \mathbf{B_r}(B)$$

Substituting the above information into ( ), we find that

$$\mathbf{B_r} = \frac{\mu I R z}{2\pi}\left(\frac{1}{\sqrt{(z^2+(R-r)^2)(\sqrt{(R-r)^2+z^2+R^2})}} - \left(\frac{1}{\sqrt{(z^2+(R+r)^2)(\sqrt{(R+r)^2+z^2+R^2})}}\right) \quad (4)$$

Following a similar line of logic as in 2.1, we have that

$$d\mathbf{B_r}(\text{Loop}) = \frac{\mu I n R z}{2\pi}\left(\frac{1}{\sqrt{(z^2+(R-r)^2)(\sqrt{(R-r)^2+z^2+R^2})}} - \left(\frac{1}{\sqrt{(z^2+(R+r)^2)(\sqrt{(R+r)^2+z^2+R^2})}}\right)dz \ (5)$$

Where d**Br(**Loop) is the infinitesimal contribution of a given dz with dN=ndz number of (square) loops.

Integration of this expression yields an approximation for **Br** at a point P a distance r radially from the axis of the loop and a distance z' from the center of the solenoid.

$$\mathbf{B_r}=\frac{\mu I n R}{2\pi}\int_{\frac{len}{2}-z'}^{\frac{Len}{2}-z'}\left(\frac{z}{\sqrt{(z^2+(R-r)^2)(\sqrt{(R-r)^2+z^2+R^2})}} - \left(\frac{z}{\sqrt{(z^2+(R+r)^2)(\sqrt{(R+r)^2+z^2+R^2})}}\right)dz \ (6)$$

Evaluating this integral yields:

Br= ln(…)

3 Magnetic Field outside a finite solenoid

In the following derivations, we will use the same cylindrical coordinate system with the standard (r,ϕ, z) coordinates. The z-axis lies on the central axis of the solenoid. ϕ is the azimuthal angle while r is measured from the center axis. r' is the radial distance from the axis of the solenoid to a point P above it.

We will derive multiple approximations depending on the relative values of r', R, and Len for a single component.

3.1 Axial component of magnetic field (r'<<Len)

Consider an arbitrary rectangular closed loop placed such that a side of length Δz lies on the axis of the solenoid. The center of the side lies an arbitrary distance z' from the center of the solenoid. The other two sides extend radially a length R+Δr. Assume that Δr~0 and Δz~0. Label the left and right corners on the axis of the solenoid a and b respectively. Label the left and right corners at r'= R+Δr e and d respectively. Lastly, label the left and right points where the loop intersect the solenoid as f and c respectively. (Needs to be cleared up)

The integral form of Ampere's Law states that $\oint \boldsymbol{B} * \boldsymbol{dT} = \mu \iint \boldsymbol{J} * \boldsymbol{dA} + \mu\varepsilon\frac{d\Psi}{dt}$, where d**T** is the infinitesimal tangent vector around a closed path, **J** is the current density, d**A** is the infinitesimal areal normal vector, Ψ is the electric flux, and ε is the vacuum permittivity. "*"denotes the inner product.

In this case, $\Psi = 0$. Since $\iint \boldsymbol{J} * \boldsymbol{dA}$ is the enclosed current, I_e the above expressions simplifies to:

$$\oint \boldsymbol{B} * \boldsymbol{dT} = \mu I_e \ (6)$$

Applying the above formula to the rectangular loop and replacing d**T** with either d**z** or d**r**, we have

$$\int_a^b \boldsymbol{B}_{in} * \boldsymbol{dz} + \int_b^c \boldsymbol{B}_{in} * \boldsymbol{dr} + \int_c^d \boldsymbol{B}_{out} * \boldsymbol{dr} + \int_d^e \boldsymbol{B}_{out} * \boldsymbol{dz} + \int_e^f \boldsymbol{B}_{out} * \boldsymbol{dr} + \int_f^a \boldsymbol{B}_{in} * \boldsymbol{dr} = \mu I_e (7)$$

Symmetry considerations tell us that **B** is a function of r and z and has two components **B**(r)$_{(r,z)}$ **and B**(z). We ignore **B**(ϕ) at this point in the paper. Therefore, we have

$$\mathbf{B}_{(r,z)}=\mathbf{B(r)}_{(r,z)}+\mathbf{B(z)}_{(r,z)} \ (8)$$

We then can replace **B** in (7) with (8). Furthermore, we can utilize the distributivity of the inner product to get

$$\int_a^b B(z)_{in}\, dz + \int_b^c B(r)_{in}\, dr + \int_c^d B(r)_{out}\, dr + \int_d^e B(z)_{out}\, dz - \int_e^f B(r)_{out}\, dr - \int_f^a B(r)_{in}\, dr = \mu I_e (9)$$

Since Δz~0, We can safely say that the $B(r)_{in}$ on the interval b-c and a-f are approximately of the same magnitude. Likewise, $B(r)_{(out)}$ on the interval c-d and e-f are approximately of the same magnitude. Thus, the integrals in question are equal in magnitude but they are opposite in sign. We then can eliminate these integrals, yielding

$$\int_a^b B(z)_{in}\, dz + \int_d^e B(z)_{out}\, dz = \mu I_e (10)$$

Since Δz~0, we can assume $B(z)$ to be roughly constant over the integration region of both integrals. Hence, we can factor B(z) out of both integrals which just become Δz. Finally, since I_e=In Δz, we have

$$B(z)_{in}\Delta z + B(z)_{out}\Delta z = \mu In\Delta z (11)$$

Substituting (5) in for $B(z)_{in}$ evaluated at z' and simplifying, we arrive at

$$B(z)_{out} = \mu mI(1 - \frac{1}{2}\left(\frac{\frac{Len}{2}-z'}{\sqrt{R^2+\left(\frac{Len}{2}+z'\right)^2}} + \frac{\frac{Len}{2}+z'}{\sqrt{(R^2+\left(\frac{Len}{2}+z'\right)^2}}\right) (12)$$

We can normalize the expression above with $K = \frac{Len}{R}$ and $C = \frac{z'}{R}$ a and to get

$$B(z)_{out} = \mu mI(1 - \frac{1}{2}\left(\frac{\frac{K}{2}-C}{\sqrt{1+\left(\frac{K}{2}-C\right)^2}} + \frac{\frac{K}{2}+C}{\sqrt{(1+\left(\frac{K}{2}+C\right)^2}}\right) (13)$$

This is only an approximation, as it assumes that both B(z) and B(r) are constant over a small Δz. Furthermore, it can only be true close to the solenoid (r~R) as the expression above does not vary with $\frac{1}{r^k}$, where k is a constant as required by the Biot-Savart law.

It is clear that $\lim_{K\to\infty} B(z)_{out}$=0, the expected result.

3.2 Axial component of magnetic field (r'>>R)

Here, we will use the spherical coordinates (r, θ, Ψ'). Θ is the polar angle, r is the radial distance from the origin, and Ψ' is the azimuthal angle.

We first explore the case of a single current loop. Here, R is the radius of the loop and r is the distance from the center of the loop to a point P that is an axial distance z and radial distance r' from the axis of the loop. We work in a coordinate system that takes z=0 at the point P. Below, we look at the scenario where z is positive.

Ref( ) states the following:

$$B_r = \frac{\mu I R^2 \cos\theta}{2r^3} (14)$$

$$B_\theta = \frac{\mu I R^2 \sin\theta}{4r^3} (15)$$

$$B_{\Psi'} = 0\ (16)$$

In this case, $r=\sqrt{(r'^2 + z^2)}$, $\cos\theta = \frac{-z}{\sqrt{r'^2+z^2}}$, and $\sin\theta = \frac{r'}{\sqrt{r'^2+z^2}}$. Substituting these values into (14) and (15) yields

$$B_r = \frac{-\mu I R^2 z}{2\left(r'^2+z^2\right)^2} (17)$$

$$B_\theta = \frac{\mu I R^2 r'}{4\left(r'^2+z^2\right)^2} (18)$$

To find the axial components of each B field component, we scale each B-field component by an appropriate trigonometric scalar. It is an easy task to determine the angles between the B field components and the z-axis. We state that (Picture?)

$$\mathbf{B_r}(z) = \mathbf{B_r}\cos(\pi - \theta) = \frac{\mu I R^2 z^2}{2\left(r'^2+z^2\right)^{\frac{5}{2}}}\,\mathbf{z}\quad (19)$$

$$\mathbf{B_\theta}(z) = \mathbf{B_\theta}\sin(\pi - \theta) = \frac{\mu I R^2 r'^2}{4\left(r'^2+z^2\right)^{\frac{5}{2}}}\,\mathbf{z}\quad (20)$$

Where $\mathbf{B_r}(z)$ and $\mathbf{B_\theta}(z)$ are the components of B_r and B_θ on the z-axis respectively. **Z** is the unit vector along the z-axis
Since, $\mathbf{B}(z) = \mathbf{B_r}(z) - \mathbf{B_\theta}(z)$, we have

$$\mathbf{B}(z)=\frac{\mu I R^2}{2(r'^2+z^2)^{\frac{5}{2}}}\left(z^2-\frac{r'^2}{2}\right)\mathbf{z} \quad (21)$$

Assuming that the solenoid is a tightly-packed collection of loops, we can say that each loop contributes a

$$dB(loop)=\frac{\mu I R^2}{2(r'^2+z^2)^{\frac{5}{2}}}\left(z^2-\frac{r'^2}{2}\right)(22)$$

to the total B-field on the axis of the solenoid.

For a given dz of the solenoid, the value of dI within that dz is given by dI=$Indz$, where $n=\frac{N}{Len}$, the number of turns (N) per unit length (Len). We then have:

$$dB(loop)=\frac{\mu I n R^2 dz}{2(r'^2+z^2)^{\frac{5}{2}}}\left(z^2-\frac{r'^2}{2}\right)(23)$$

Integration of this expression over the solenoid yields the total Bnet$_z$ component at any point P with z-coordinate z' (measured from the center of the solenoid). This integral is easily solved by performing a standard trigonometric substitution with $z = R\tan\text{ø}$.

$$B_z=\int_{-\frac{Len}{2}-z'}^{\frac{Len}{2}-z'}\frac{\mu I n R^2 dz}{2(r'^2+z^2)^{\frac{5}{2}}}\left(z^2-\frac{r'^2}{2}\right)=\mu m I R^2\left(\frac{2z'-Len}{((2z'-Len)^2+4r'^2)^{\frac{3}{2}}}-\frac{2z'+Len}{((2z'+Len)^2+4r'^2)^{\frac{3}{2}}}\right)(24)$$

Again, this is only an approximation for a finite r' as the above equation is only an approximation for finite r' as it assumes that the solenoid is an infinitely tightly-packed stack of closed loops.

It is clear that (keeping r' and z' constant) $\lim_{Len\to\infty} B(z)_{out}$=0, the expected result.

3.3 Radial component of magnetic field (r'>>R)

We repeat the same method as in 3.2. To find the radial components of each B field component, we scale each B-field component by an appropriate trigonometric scalar. It is an easy task to determine the angles between the B field components and the z-axis. We state that, for +z, (Picture?)

$$\mathbf{B_r}(r)=\mathbf{B_r}\sin(\pi-\theta)=\frac{\mu I R^2 z' r'}{2(r'^2+z^2)^{\frac{5}{2}}}\boldsymbol{r} \quad (25)$$

$$\mathbf{B_\theta}(r)=\mathbf{B_\theta}\cos(\pi-\theta)=\frac{\mu I R^2 r'^1 z'}{4(r'^2+z^2)^{\frac{5}{2}}}\boldsymbol{r} \quad (26)$$

Where $\mathbf{B_r}$(r) and $\mathbf{B_\theta}$ (r) are the components of $\mathbf{B_r}$ **and** $\mathbf{B_\theta}$ in the radial direction respectively. **R** is the unit vector in the radial direction

Since, **B(**r)= $\mathbf{B_r}$(r)- $\mathbf{B_\theta}$ (r), we have

$$\mathbf{B}(r)=\frac{-3\mu I R^2 r' z'}{3(r'^2+z^2)^{\frac{5}{2}}}\mathbf{z} \quad (27)$$

For -z, B(r) becomes positive. In integrating the above expression in the same way outlined above in this paper, we have to be careful regarding this sign change.

$$B(r)=\int_{-\frac{len}{2}-z'}^{0}\frac{z}{(z^2+r'^2)^{5/2}}dz-\int_{0}^{\frac{Len}{2}-z'}\frac{z}{(z^2+r'^2)^{\frac{5}{2}}}dz \ (28)$$

Finding the antiderivative is easy, simply let u=$z^2+r'^2$. Evaluating the definite integrals yields:

$$B(r)=\frac{2}{3}\left(\frac{1}{\left(\frac{len}{2}-z'\right)^2+r'^2)^{\frac{3}{2}}}-\frac{1}{r'^3}\right). \quad (29)$$

It is clear that the above equation is only an approximation for finite r' as it assumes that the solenoid is an infinitely tightly-packed stack of closed loops.

Clearly, (Keeping r' and z' constant) $\lim_{Len\to\infty} B(r)_{out}$=0, the expected result.

3.3 Azimuthal Component of Magnetic Field

A real solenoid is not simply a collection of loops. Rather, there is an axial current, I_a that runs along the length of the solenoid. Hence, there will be a B(φ) component outside the solenoid. We shall treat the solenoid as a cylindrical sheet of current I_a (radius R, length Len) and curent density J. We shall use the cylindrical coordinates (r, φ, z)

We can view this sheet of current as made up of infinitely many thin “strips” of current with length Len and width Rdφ.

The B-field of a finite wire is known:

$$B(wire) = \frac{\mu I}{4\pi r}\left(\frac{Len + 2z'}{\sqrt{4r^2 + (Len + 2z')^2}} + \left(\frac{Len - 2z')}{\sqrt{4r^2 + (Len - 2z')^2}}\right) \quad (30)$$

Where r is the radial distance from the wire to an arbitrary point p and z’ is the distance from the center of the wire to P. r’ denotes the perpendicular distance from P to the axis of the solenoid. For a single “strip”, we can use the above formula to calculate the contribution of that wire to the total B-field.

Simple geometry and the law of cosines tells us that for an arbitrary “strip”, $r = \sqrt{R^2 + r'^2 - 2Rr\sin\varphi}$. We the have:

$$B(wire) = \frac{\mu I}{4\pi\sqrt{R^2+r'^2-2Rr\sin\varphi}}\left(\frac{Len+2z'}{\sqrt{4(R^2+r'^2-2Rr\sin\varphi)+(Len+2z')^2}} + \left(\frac{Len-2z')}{\sqrt{4(R^2+r'^2-2Rr\sin\varphi)+(Len-2z')^2}}\right) \quad (31)$$

The net B-field only has a φ-component, and each wire contributes to some portion of that B-field. Vector components yields that **B**(φ)=**B**$|\cos\varphi|$. The absolute value bars are added as $\cos\varphi$ is negative for $\frac{\pi}{2} < \varphi < \pi$.

Furthermore, we know that $\frac{dI}{Len R d\varphi} = \frac{I_a}{2\pi Len R} = J$, where dI is the infinitesimal amount of current in each strip.

Solving for dI in terms of I_a and writing B(φ) in terms of an integral yields

$$B(\varphi) = \frac{\mu I_a}{8\pi^2}\int_0^{2\pi} \frac{|\cos\varphi|}{(R^2 + r'^2 - 2Rr\sin\varphi)}\left(\frac{Len + 2z'}{\sqrt{4(R^2 + r'^2 - 2Rr\sin\varphi) + (Len + 2z')^2}} + \frac{Len - 2z')}{\sqrt{4(R^2 + r'^2 - 2Rr\sin\varphi) + (Len - 2z')^2}}\, d\varphi \quad (32)$$

This integral can be evaluated numerically. The exact value of I_a depends on the geometry of the solenoid. Of course, this expression is only an approximation because it assumes that the solenoid is a cylindrical sheet of current I_a, which is not the case, no matter the exact geometry of the solenoid.

4. Conclusion

We have presented 7 quantitative expressions for the B-field of the finite solenoid. These can be used in practice or for teaching purposes regarding magnetic fields in general

5. References

1) Y +F
2) Giancoli
3) NASA Paper